Resumen del trabajo candidato al premio Ramiro Melendreras

Aida Calviño Martnez
Department of Statistics and Operations Research III
Complutense University of Madrid, Madrid, Spain

Abstract

Accelerated life-testing (ALT) is a very useful technique for examining the reliability of highly reliable products. It allows testing the products at higher than usual stress conditions to induce failures more quickly and economically than under typical conditions. A special case of ALT are step-stress tests that allow experimenter to increase the stress levels at fixed times. This paper deals with the multiple step step-stress model under the cumulative exposure model with lognormally distributed lifetimes in the presence of Type-II and Progressive Type-II censoring. For this model, the maximum likelihood estimates (MLE) of its parameters, as well as the corresponding observed Fisher Information Matrix (FI), are derived. The likelihood equations do not lead to closed-form expressions for the MLE, and they need to be solved by means of an iterative procedure, such as the Newton-Raphson method. We then evaluate the bias and mean square error of the estimates and provide asymptotic and bootstrap confidence intervals. Finally, in order to asses the performance of the confidence intervals, a Monte Carlo simulation study is conducted.

State of the art

Nowadays, most manufactured products are highly reliable with large lifetimes that result in large costs and high experimental times when testing them under typical conditions. In those cases, when conventional life-testing becomes unuseful, the reliability experimenter may adopt accelerated life-testing, wherein the experimental units are subjected to higher stress levels than under normal operating conditions. Accelerated life tests (ALT) are used to quickly obtain information on the life time distribution of products by testing them at higher than nominal levels of stress to induce early failures. Furthermore, ALTs allow to examine the effect of stress factors, such as pressure or temperature, on the lifetimes of experimental units. Data collected from ALT needs to be fitted to a model that relates the lifetime to stress and estimate the parameters of the lifetime distribution under normal conditions. This requires a model to relate the levels of stress to the parameters of the lifetime distribution. One such model is the *cumulative exposure model* introduced by Sedyakin (1966).

Accelerated life-testing may be performed either at increasing or constant high stress levels. In practice, constant stress ALT leads to very few failures within the experimental time, reducing the effectiveness of accelerated testing. A particular case of accelerated testing is the *step-stress model*, which allows for a change of stress in steps at various intermediate stages of the experiment. Specifically, a random sample of n units is placed on a life test at an initial stress level x_1. At prefixed times $\tau_1, \tau_2, \ldots, \tau_{m-1}$, the stress levels are increased to $x_2, x_3, \ldots, x_m$, respectively.

The step-stress model has been discussed extensively in the literature. Ganguly et al. (2015) and Xiong (1998), among others, have all considered inference for the step-stress model assuming exponential lifetimes based on different censoring schemes. Under progressive Type-II censoring, Xie et al. (2009) developed both inference and optimal progressive scheme. While all these discussions deal with exponential step-stress models, Khamis and Higgins (1998) and Kateri and Balakrishnan (2008) assume Weibull distributed lifetimes. Balakrishnan et al. (2009) and Lin and Chou (2012) have developed simple and multiple step-stress models, respectively, with lognormally distributed lifetimes and Type-I censoring. For a comprehensive review on step-stress models refer to Gouno and Balakrishnan (2001)or Balakrishnan (2009). One may refer to Balakrishnan (2007) for a overview of various developments relating to progressive censoring.

Main contributions

In this paper, we assume both Type-II censoring, where the experiment terminates when a pre-specified number r ($r < n$) of failures is observed; and **Progressive Type-II censoring**, where a predetermined number of survival units is removed from the test whenever a failure occurs and, again, the experiment terminates when a pre-specified number of failures is reached. Progressive censoring schemes are not very common in the literature. However, they are a very useful tool as they permit obtaining information about the lifetimes of the units without the need of exposing all units to high levels of stress[1], and therefore, with lower associated costs.

We further assume **lognormally distributed life times**. As it can be deduced from the previous section, the literature on this topic is mainly focused on the exponential distribution. However, it is important to develop inference methods for other types of distributions in order to take into account the different lifetime distributions that may arise in real life.

Moreover, instead of the simple step-stress model commonly assumed in the literature, we develop a **multiple step-stress model** where the number of steps can be higher than two (and, thus, has as a special case the simple model). The location parameters μ_i of the lognormally distributed lifetimes in each step are given by the linear link function: $\mu_i = \gamma_0 + \gamma_1 x_i$. Assuming a link function permits avoiding estimating one location parameter per step, as only γ_0 and γ_1 need to be estimated independently of m and, therefore, there is no need to impose a minimun number of failures at each step. Furthermore, it allows obtaining the parameters of the lognormal life time distribution for whatever lever of stress x_i. When assuming this link function, some physical models, such as the Arrhenius equation or the Inverse Power relationship, can be applied, which permits modeling more real situations.

In summary, let T_i and x_i be the lifetime and stress level at step i. Then,

$$\begin{aligned}
&\log(T_i) = \gamma_0 + \gamma_1 x_i + \epsilon, \quad \epsilon \sim N(0, \sigma^2) \\
&E[\log(T_i)] = \gamma_0 + \gamma_1 x_i = \mu_i \\
&\log(T_i) \sim N(\mu_i, \sigma^2),
\end{aligned}$$

where we further assume that the scale parameter σ is free of the stress levels.

Without loss of generality, we briefly introduce the multiple step-stress testing experiment under progressive Type-II censoring (note that classical Type-II censoring is a special case of progressive Type-II censoring, where all the units are removed from the experiment when the r-th failure is observed).

Let $x_1, x_2, \ldots, x_m$ be the stress levels, such that $x_1 < x_2 < \cdots < x_m$. A random sample of n experimental units are placed on a life-test at an initial stress level of x_1. Then, at a prefixed time τ_1, the stress level is changed to x_2; next, at time τ_2, the stress level is changed to x_3, and so on. At the time of the first failure, R_1 of the $n-1$ surviving units are randomly removed from the experiment; at the time of the second failure, R_2 of the $n-2-R_1$ surviving units are randomly removed from the experiment, and so on; the test continues until the r^{th} failure occurs at which time all the remaining $R_r = n - r - R_1 - \cdots - R_{r-1}$ surviving units are removed.

For $i = 1, 2, \ldots, m$, let n_i be the number of units failed at stress level x_i (i.e., in the time interval $(\tau_{i-1}, \tau_i]$), and $t_{i,j}$ denote the jth ordered failure time out of n_i units at level x_i, $j = 1, 2, \ldots n_i$.

At stress level x_i, the life time T_i of a test unit is assumed to follow a lognormal distribution with location and scale parameter μ_i and σ, respectively. The location parameters μ_i are given by the following **linear link function**:

$$\mu_i = \mu(x_i) = \gamma_0 + \gamma_1 x_i, \tag{1}$$

and the scale parameter σ is assumed to be free of the stress levels. Therefore, we need to estimate the regression parameters γ_0 and γ_1 (that will give us the relation between the life time and the stress level) as well as the scale parameter.

[1]Note that, in this case, the units are censored before the end of the experiment and, thus, are not exposed to the same level of stress than the last failed unit. For that reason, their residual life is larger and can be reused more easily.

We further assume that the data comes from a **cumulative exposure model**. This model relates the lifetime distribution of experimental units at one stress level to the distributions at preceding stress levels by assuming that the residual life (i.e., the survival/reliability function) of the experimental units depends only on the cumulative exposure the units have experienced, with no memory of how this exposure was accumulated. This is obtained by adding an artificial extra time (s_{i-1}) to the lifetimes at each stress level (i) to reflect the exposure suffered at previous levels. As the exposure is reflected in the survival probability, s_{i-1} is given by the equation

$$F_i(s_{i-1};\mu_i;\sigma) = F_{i-1}(\tau_{i-1} + s_{i-2} - \tau_{i-2};\mu_{i-1},\sigma), \quad i = 2,3,\ldots,m, \tag{2}$$

where $\tau_0 = 0$, $s_0 = 0$, and $F_i(t;\mu_i,\sigma)$ is given the Cumulative Distribution Function (CDF) of a log-normal distribution with location and scale parameters equal to μ_i and σ, respectively. Note that s_{i-1} is an artificial time added equivalent to the "damage" suffered in the previous steps, in terms of the CDF.

The **likelihood function** for the parameter vector $\boldsymbol{\theta} = (\gamma_0, \gamma_1, \sigma)$ based on the observed data and the previous assumptions is

$$L(\boldsymbol{\theta}) = C \prod_{i=1}^{m} \left\{ \prod_{j=1}^{n_i} g(t_{i,j}) \left[1 - G(t_{i,j})\right]^{R_{k(i,j)}} \right\}, \tag{3}$$

where g and G and the pdf and CDF of the lifetimes under the Cumulative exposure model and log-normal distribution previously defined, $k(i,j)$ refers to the ordered position of the (i,j)-th unit in the global sample and C is a constant (given in the paper). This likelihood takes into account the information given by the censored units: its lifetime is, at least, equal to the time they were part of the experiment.

From Equation (3, we further determine the likelihood equations for the parameters. However, explicit solutions do not exist for them and **numerical methods**, such as the Newton-Raphson procedure, need to be used to compute the MLE of $\boldsymbol{\theta}$.

The Fisher information matrix, which is the inverse of the variance-covariance matrix of the MLE of the vector parameter, is also given to be used later in the computation of the **approximate confidence intervals**.

Bootstrap confidence intervals, in particular Percentile Bootstrap CIs, are also derived. As a requirement of the Bootstrap CIs computation, an algorithm to generate Bootstrap samples of this complex model (multiple step-stress with progressive Type-II censoring) is also given.

A **Monte Carlo simulation study** has been conducted in order to evaluate the performance of the proposed method. It includes several sample sizes, censoring proportions, censoring schemes, number of stress levels, variance levels, etc. The results include bias, Mean Square Errors, coverage probabilities and lengths of the two types of confidence intervals proposed. They are shown by means of tables and figures for a better understanding and analysis.

It is shown that the method provides accurate and almost unbiased estimates, as well as well-performing confidence intervals. Better results are obtained when the number of steps in the model is large and complete right censoring is avoided. As usual, the larger number of non-censored units, the more precise are the results. An interesting feature of this model is the fact that large variances do not affect the quality of the results: similar levels of accuracy and interval lengths.

References

Balakrishnan, N. (2007). Progressive censoring methodology: an appraisal. *Test*, 16(2):211–259.

Balakrishnan, N. (2009). A synthesis of exact inferential results for exponential step-stress models and associated optimal accelerated life-tests. *Metrika*, 69:351–396.

Balakrishnan, N., Zhang, L., and Xie, Q. (2009). Inference for a simple step-stress model with Type-I censoring and lognormally distributed lifetimes. *Communications in Statistics, Theory and Methods*, 38:1690–1709.

Ganguly, A., Kundu, D., and Mitra, S. (2015). Bayesian analysis of a simple step-stress model under weibull lifetimes. *IEEE Transactions on Reliability*, 64(1):473–485.

Gouno, E. and Balakrishnan, N. (2001). Step-stress accelerated life test. In N. Balakrishnan and C. Rao, eds., *Advances in Reliability*, vol. 20 of *Handbook of Statistics*, pp. 623–639. North-Holland, Amsterdam.

Kateri, M. and Balakrishnan, N. (2008). Inference for a simple step-stress model with Type-II censoring, and weibull distributed lifetimes. *IEEE Trans. Reliability*, 57:616–626.

Khamis, I. H. and Higgins, J. J. (1998). A new model for step-stress testing. *IEEE Trans. Reliability*, 47:131–134.

Lin, C. and Chou, C. (2012). Statistical inference for a lognormal step-stress model with Type-I censoring. *IEEE Trans. Reliability*, 61:361–377.

Sedyakin, N. M. (1966). On one physical principle in reliability theory (in Russian). *Techn. Cybernet.*, 3:80–87.

Xie, Q., Balakrishnan, N., and Han, D. (2009). Advances in Mathematical and Statistical modeling. In B. C. Arnold, N. Balakrishnan, J. M. Sarabia, and R. Mínguez, eds., *Exact Inference and Optimal Censoring Scheme for a Simple Step-Stress Model Under Progressive Type-II Censoring*, pp. 107–137. Birkhauser, Berlin.

Xiong, C. (1998). Inference on a simple step-stress model with Type-II censored exponential data. *IEEE Trans. Reliability*, 47:142–146.